Obtaining Nonlinear Aerodynamic Stability Coefficients from Free-Angular Motion of Rigid Bodies

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An extension of the method of Kryloff and Bogoliuboff is employed to obtain an approximate solution of the nonlinear differential equations of motion for the complex angle of attack. The primary objective is to obtain, from the angular motions of rigid bodies, the nonlinear restoring, damping, and Magnus moment stability coefficients. Using representative values of these coefficients and initial conditions, angular motions are simulated by numerical integration of the complete equations of motion. A nonlinear data reduction procedure is applied to the integrated data in order to evaluate the nonlinear theory. This evaluation indicates that the nonlinear aerodynamic stability coefficients could be accurately determined. A three-degree-of-freedom subsonic wind tunnel testing technique has been used to obtain experimental angular data for an Apache sounding rocket model. From the application of the nonlinear theory to the wind-tunnel data, it is shown that the aerodynamic stability coefficients are highly nonlinear functions of angle of attack.

Nomenclature

A = reference area $A = \pi d^2/4$
d = reference length, body diam, ft
e = percentage error in fitted value relative to input value
e_t = error in total coefficient
$\mathbf{F}$ = $\begin{vmatrix} F_x \\ F_y \\ F_z \end{vmatrix}$, forces along x,y,z axes, respectively
I = pitching moment of inertia, slug-ft²
I_x = rolling moment of inertia, slug-ft²
$K_{N,P}$ = nutation and precession model amplitudes, deg
K_R,K_T = repose and trim amplitudes, respectively, deg
$\mathbf{M}$ = $\begin{vmatrix} L \\ M \\ N \end{vmatrix}$, moment about body axes, respectively
p,q,r = angular velocities about body axes, respectively, rad/sec
Q = dynamic pressure
s = gyroscopic stability factor
u,v,w = linear velocities along body axes, respectively, fps
v = magnitude of velocity vector, fps
$\boldsymbol{\alpha}$ = complex angle of attack, $\boldsymbol{\alpha} = \beta + i\alpha$, deg
α = angle of attack, deg
$|\boldsymbol{\alpha}|^2$ = magnitude of complex angle of attack squared, $|\boldsymbol{\alpha}|^2 = \boldsymbol{\alpha}\ \boldsymbol{\alpha}_c$, $\boldsymbol{\alpha}_c$ complex conjugate of α, deg²
β = angle of side-slip, deg
$\lambda^*_{N,P}$ = uncorrected nutation and procession damping rates, rad/sec
$\lambda^v_{N,P}$ = nutation and precession damping rates caused by variable frequency, rad/sec
$\lambda_{N,P}$ = corrected nutation and precession damping rates, rad/sec
ρ = density, slug/ft³
$\omega_{N,P}$ = nutation and precession frequencies (rad/sec)
$C_{M\alpha}(|\boldsymbol{\alpha}|)$ = pitching moment stability coefficient, $C_{M\alpha_2} + C_{M\alpha_2}|\boldsymbol{\alpha}|^2 \equiv (M_{\alpha_0} + M_{\alpha_2}|\boldsymbol{\alpha}|^2)/QAd$, rad⁻¹
$C_{Mq}(|\boldsymbol{\alpha}|) + C_{M\dot{\alpha}}(|\boldsymbol{\alpha}|)$ = damping moment stability coefficietn, $= (C_{Mq} + C_{M\dot{\alpha}})_0 + (C_{Mq} + C_{M\dot{\alpha}})_2|\boldsymbol{\alpha}|^2 \equiv [(M_q + M_{\dot{\alpha}})_0 + (M_q + M_{\dot{\alpha}})_2|\boldsymbol{\alpha}|^2]/(QAd^2/2V)$, rad⁻¹
$C_{M_{p\alpha}}(|\boldsymbol{\alpha}|)$ = Magnus moment stability coefficient, $= C_{M_{p\alpha_0}} + C_{M_{p\alpha_2}}|\boldsymbol{\alpha}|^2 \equiv (M_{p\alpha_0} + M_{p\alpha_2}|\boldsymbol{\alpha}|^2)/(QAd^2/2V)$, rad⁻²

Subscripts

0 = linear term of polynomial expansion
2 = quadratic term of polynomial expansion
N,P = nutation and precession modes

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Introduction

AS wind-tunnel testing techniques for the evaluation of the dynamic behavior of finned missiles and projectiles have improved, additional nonlinear theories and data reduction procedures are needed. This work[1] was initiated to develop a theory and a data reduction procedure for determining the nonlinear aerodynamic stability coefficients from the angular motions of rigid bodies.

The linear aeroballistic theory advanced by Fowler et al.,[2] in 1920, was clarified and extended by Kent,[3,4] Neilson and Synge,[5] and McShane, Kelly, and Reno.[6,7] More recently Nicolaides[8] and Murphy[9,10] included effects of nonlinearities in the aerodynamic stability coefficients with angle of attack. In the quasi-linear theories of Refs. 8 and 10, the effects of a nonlinear Magnus moment were examined. However, Ref. 8 did not include the nonlinearity of the damping and restoring moments. The nonlinear theory of Ref. 10 dealt with nonlinear restoring, damping and Magnus moments, but the nonlinear damping moment was not considered for experimentation.

In this paper, a theory and data reduction procedure,[1] which includes the nonlinear variations of the restoring, damping and Magnus moments with angle of attack, are applied to the constrained angular oscillations of a fin missile. First, to evaluate the theory, representative values of the nonlinear aerodynamic stability coefficients are used to integrate numerically the complete equations of motion on the UNIVAC 1107 digital computer. A six-degree-of-freedom trajectory simulation computer program[16,17] is used to determine angle-of-attack time histories. Following this evaluation, the theory and data reduction procedure are applied to experimental data for the subsonic angular motions of an Apache sounding rocket model. The constrained three-degree-of-freedom wind-tunnel testing procedure employs a jewel bearing support system to minimize nonaerodynamic friction.

Linear Theory

For a missile with mirror symmetry and trigonal or greater rotational symmetry, the aerodynamic moments are assumed to be linear functions of angle of attack as follows:

$$M + iN = -iM_\alpha \boldsymbol{\alpha} + M_q q - iM_{\dot\alpha}\dot{\boldsymbol{\alpha}} - pM_{p\alpha}\boldsymbol{\alpha} - iM_{\delta_\epsilon}\delta_\epsilon e^{ipt} \quad (1)$$

Assuming constant roll rate, velocity, and altitude, the differential equation for the complex angular motion is given as

$$\ddot{\boldsymbol{\alpha}} - [ipI_x/I + (M_q + M_{\dot\alpha})/I]\dot{\boldsymbol{\alpha}} - (M_\alpha/I + ipM_{p\alpha}/I)\boldsymbol{\alpha} = M_{\delta_\epsilon}\delta_\epsilon e^{ipt} + \mathbf{K}_R \quad (2)$$

The solution of this differential equation as given in Ref. 8 is

$$\boldsymbol{\alpha} = \mathbf{K}_{N}e(\lambda_N + i\omega_N)t + \mathbf{K}_P e(\lambda_p + i\omega_p)t + \mathbf{K}_T e^{ipt} + \mathbf{K}_R \quad (3)$$

where

$$\lambda_{N,P} = \frac{QAd^2}{2V}\left[\frac{C_{z_\alpha}}{md^2}(1 \mp \tau) + \frac{(C_{M_q} + C_{M\dot\alpha})}{2I}(1 \pm \tau) \pm C_{M_{p\alpha}}\frac{\tau}{I_x}\right] \quad (4)$$

$$\omega_{N,P} = (1 \pm \tau^{-1})pI_x/2I \quad (5)$$

$$\tau = (1 - s^{-1})^{-1/2} \quad (6)$$

$$s = (pI_x)^2/4IC_{M\alpha}QAd \quad (7)$$

Nonlinear Theory

For a missile with mirror symmetry and trigonal or greater rotational symmetry, the aerodynamic moments are considered to be nonlinear functions of $|\boldsymbol{\alpha}|^2$, as specified in Refs. 12 and 13. This functional relation is

$$-i(M + iN) = M_\alpha(|\boldsymbol{\alpha}|)\boldsymbol{\alpha} + M_q(|\boldsymbol{\alpha}|)\boldsymbol{\alpha} + M_{\dot\alpha}(|\boldsymbol{\alpha}|)\dot{\boldsymbol{\alpha}} + iM_{p\alpha}(|\boldsymbol{\alpha}|)p\boldsymbol{\alpha}$$

where

$$\begin{aligned} M_\alpha(|\boldsymbol{\alpha}|) &= M_{\alpha_0} + M_{\alpha_2}|\boldsymbol{\alpha}|^2 \\ M_q(|\boldsymbol{\alpha}|) &= M_{q_0} + M_{q_2}|\boldsymbol{\alpha}|^2 \\ M_{\dot\alpha}(|\boldsymbol{\alpha}|) &= M_{\dot\alpha_0} + M_{\dot\alpha_2}|\boldsymbol{\alpha}|^2 \\ M_{p\alpha}(|\boldsymbol{\alpha}|) &= M_{p\alpha_0} + M_{p\alpha_2}|\boldsymbol{\alpha}|^2 \end{aligned} \quad (8)$$

Assuming constant roll rate, velocity and altitude, the differential equation for the complex angular motion becomes,

$$\ddot\alpha - [iPI_x/I + (M_q + M_{\dot\alpha})_0/I + (M_q + M_{\dot\alpha})_2/I|\boldsymbol{\alpha}|^2]\dot{\boldsymbol{\alpha}} - [M_{\alpha_0}/I + M_{\alpha_2}/I|\boldsymbol{\alpha}|^2 + iP/I(M_{p_{\alpha_0}} + M_{p_{\alpha_2}}|\boldsymbol{\alpha}|^2)]\boldsymbol{\alpha} = 0 \quad (9)$$

The motion described by this nonlinear differential equation is oscillatory and must, to some extent, conform to that of the linear case, Eq. (3). Hence, a solution of the nonlinear differential equation should have a form similar to that of the linear. Therefore, an approximate solution of the nonlinear differential equation, Eq. (9), using the method of Refs. 1, 9, 10 and 14, is given by

$$\boldsymbol{\alpha} = \beta + i\alpha = K_N e^{i\phi_N} + K_p e^{i\phi_p} \quad (10)$$

$$K_N = K_{N_0}e^{\lambda^*_N t},\ K_P = K_{P_0}e^{\lambda^*_P t} \quad (11)$$

$$\dot K_{N,P}/K_{N,P} \equiv \lambda^*_{N,P} = \lambda_{N,P} + \lambda_{N,P^v} \quad (12)$$

$$\lambda_{N,P^v} = [C_{M\alpha_2}\tau_{N,P}{}^2(K_{N,P}\dot K_{P,N})2IV/p^2I_x{}^2d]QAd^2/2V \quad (13)$$

$$\begin{aligned} \lambda_{N,P} = \{ & (C_{M_q} + C_{M\dot\alpha})_0(1 \pm \tau_{N,P})/2I + \\ & (C_{M_q} + C_{M\dot\alpha})_2[(K_{N,P}{}^2 + K_{P,N}{}^2)(1 \pm \tau_{N,P}) \\ & K_{P,N}{}^2\tau_{N,P}(-1/\tau_{P,N} \pm 1)]/2I \pm \\ & (C_{M_{p\alpha_0}} + C_{M_{p\alpha_2}}\delta_{\epsilon N,P}{}^2)\tau_{N,P}/I_x\}QAd^2/2V \end{aligned} \quad (14)$$

$$\phi_{N,P} = (1 \pm \tau_{N,P}{}^{-1})tpI_x/2I + \phi_{N_0P_0} \quad (15)$$

$$\omega_{N,P} = \dot\phi_{N,P}\,(1 \pm \tau_{N,P}{}^{-1})pI_x/2I \quad (16)$$

$$\tau_{N,P} = (1 - 1/s_{N,P})^{-1/2} \quad (17)$$

$$s_{N,P} = (pI_x)^2/[4IQAd(C_{M\alpha_0} + C_{M\alpha_2}\delta^2{}_{\epsilon N,P})] \quad (18)$$

$$\delta^2{}_{\epsilon N,P} = K_{N,P}{}^2 + 2K_{P,N}{}^2 \quad (19)$$

Implications of the Nonlinear Theory

It is important to note [Eq. (18)] that the effect of a nonlinear restoring moment is to cause a variable frequency. The varying frequency, as seen in Eq. (12), has a damping effect on the angular motion. As a result, the time rate of change of the modal amplitudes $K_{N,P}$ in this term cannot be replaced by the damping rates, $\lambda_{N,P}$, as was done in Ref. 9. Therefore, if the nonlinear term of the damping coefficient (C_{M_q} + $C_{M\dot\alpha_2}$) is to be determined accurately, the entire term, designated $\lambda_i{}^v$, must be considered for analysis.

Equation (13) indicates that the nonlinear restoring moment term M_{α_2} affects the damping rates and must be accounted for in the computation of the damping and Magnus coefficients. Furthermore, the damping rates for the nonlinear case are functions of the modal amplitudes. This is particularly obvious in the case of the nonlinear damping moment. The uncorrected damping rates $\lambda_{N,P}{}^*$ simplified for a slowly spinning finned missile, are given by

$$\begin{aligned} \lambda_{N,P}{}^* = [& C_{M\alpha_2}\tau_{N,P}{}^2(K_{N,P}\dot K_{N,P} + 2K_{P,N}\dot K_{P,N})2IV/p^2I_x{}^2d + \\ & (C_{M_q} + C_{M\dot\alpha})_0(1 \pm \tau_{N,P})/2I + \\ & (C_{M_q} + C_{M\dot\alpha})_2K_{N,P}{}^2(1 \pm \tau_{N,P})/2I + \\ & (C_{M_{p\alpha_0}} + C_{M_{p\alpha_2}}\delta_\epsilon{}^2)\tau_{N,P}/I_x]QAd^2/2V \end{aligned} \quad (20)$$

The interaction of the modal amplitudes and nonlinear damping moment, as well as of the nonlinear Magnus moment terms, is evident. It is important to note that a dynamic instability may be overlooked in a free-flight test, if the missile is launched with only one of the modal amplitudes initially excited. To avoid this situation, it is imperative that the nonlinearity in $|\boldsymbol{\alpha}|$ of the stability coefficients be determined. For M_{α_2}, $(M_q + M_{\alpha_2}) = 0$, Eq. (12) reduces to

$$\lambda_{N,P}{}^* = (M_q + M_{\dot\alpha})(1 \pm \tau)/2I \pm (M_{P\alpha_0} + M_{P\alpha_2}|\boldsymbol{\alpha}|^2)\tau/I_x$$

$$|\boldsymbol{\alpha}|^2 = K_N{}^2 + K_P{}^2 + 2\cos(\phi_N + \phi_P)$$

$$\delta_{\epsilon N,P}{}^2 = K_{N,P}{}^2 + 2K_{P,N}{}^2$$

Reference 8 restricts its applicability to circular motions only; hence K_N or $K_P = 0$. For $K_N \neq 0$ and $K_P = 0$,

$$\lambda_{N,P}{}^* = (\lambda_{N,P}{}^*)\ [\text{Ref. 8}] = (M_q + M_{\dot\alpha})(1 \pm \tau)/2I \pm (M_{P\alpha_0} + M_{P\alpha_2}K_N{}^2)\tau/I_x$$

The same results hold for $K_P \neq 0$ and $K_N = 0$. Therefore, for pure circular motions the theory of Ref. 8 agrees with the present development.

Data Reduction Procedure

The epicyclic equation, Eq. (10), is fitted to the angular oscillations by the method of differential corrections, using the wobble computer program.[15] By employing overlapping sectional fits of small segments of data, the stability parameters are determined as functions of time. If it is assumed that the stability parameters are relatively constant over each sectional fit, then the variation of each parameter with time contains the nonlinearity of the associated aerodynamic stability coefficient with angle of attack. The nonlinear aerodynamic stability coefficients $C_{M_\alpha}(|\alpha|)$, $C_{M_q}(|\alpha|) + C_{M_{\dot\alpha}}(|\alpha|)$ and $C_{M_{p\alpha}}(|\alpha|)$ are computed as polynomial functions of angle of attack from the stability parameters as follows: The product of ω_N and ω_P determined from Eq. (16) yields

$$\omega_N\omega_P = (C_{M_\alpha}/I + C_{M_{\alpha_2}}/I\delta_\alpha^2)QAd$$

where

$$\delta_\alpha^2 = K_N^2 + K_P^2 + (K_N^2\omega_N - K_P\omega_P)(\omega_N - \omega_P)^{-1}$$

Using a least squares technique to fit a straight line to $\omega_N\omega_P$ vs δ_α^2 yields $C_{M_{\alpha_0}}$ as the intercept and $C_{M_{\alpha_2}}$ as the slope.

The variable frequency term λ_i^v is determined as follows. From Refs. 8 and 15, it was shown that the varying frequency effects can be treated as a time-varying problem. The modal amplitudes are then given by

$$K_j(t) = K_j(0)[\omega_j(0)\omega_j(t)]^{1/2}e^{\lambda_j t}\cos[\eta_j(t) + \delta_j]$$

where

$$\eta_j = \int_0^t \omega_j(t)dt$$

and δ_j = phase angle. Correcting λ_N^* and λ_P^* by determining λ_N^v and λ_P^v from a logarithmic technique developed in Ref. 15, the actual damping rates λ_N and λ_P, Eq. (14), are fitted simultaneously with a least squares procedure to yield $(C_{M_q} + C_{M_{\dot\alpha_0}})$, $(C_{M_q} + C_{M_{\dot\alpha_2}})$, $C_{M_{p\alpha_0}}$ and $C_{M_{p\alpha_2}}$. The sign convention used for the coefficients is shown in Fig. 1.

Evaluation of the Nonlinear Theory

Using representative values of the nonlinear stability coefficients for the Apache rocket, expressed in the polynomial form of Eq. (8), the complete equations of motion were integrated numerically using a six-degree-of-freedom trajectory simulation program[16,17] to obtain α and β as functions of time. The wobble program was then used to fit Eq. (10) to the angular coordinates. Using the aforementioned data reduction procedure, the nonlinear stability coefficients were determined for comparison with the 6-D input. For this evaluation a basic set of aerodynamic stability coefficients was established for the Apache sounding rocket from 3-D constrained angular oscillation tests. Using these coefficients, wind-tunnel simulations were carried out on the computer.

Basic Aerodynamic Data

Using the basic aerodynamic data and mass parameters, a numerical integration was performed with an initial angle of attack of 12° and velocity of 34.88 ft/sec. The α and β data for this computation are presented in Fig. 2. The wobble program was used to fit Eq. (10) to these data, using 1.8 cycles of the angular motion in sections containing 190 points/cycle.

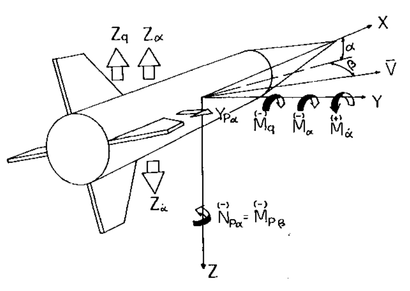

Fig. 1 Aerodynamic forces and moments acting on finned missile.

Table 1 Coefficient sectional fits for basic data set

T	$C_{M_{\alpha_0}}$	$C_{M_{\alpha_2}}$	$(C_{M_q} + C_{M_{\dot\alpha_0}})$	$(C_{M_q} + C_{M_{\dot\alpha_2}})$	$C_{M_{p\alpha_0}}$	$C_{M_{p\alpha_2}}$
3.4	−412.6	1103	−10430	−167800	−618.9	6162
3.6	−412.6	1097	−10410	−175200	−621.8	6529
3.8	−413.1	1151	−10430	−168700	−618.8	6324
6.0	−413.0	1161	−10450	−153600	−619.0	6071
6.2	−412.8	1110	−10440	−160300	−619.8	6088
6.4	−412.6	1062	−10430	−167100	−619.8	6236
Mean	−412.9	1126	−10430	−166700	−619.8	6260

From these fitted data, $K_{N,P}$, $\lambda_{N,P}^*$ and $\omega_{N,P}$ (Fig. 3), in addition to the probable error of fit of the theory to the data (Fig. 2), were obtained as functions of time. The probable error was small. The variations of the stability parameters with time verify the presence of nonlinear aerodynamic coefficients.

As pointed out in Eq. (12), the nonlinear restoring moment causes an additional damping effect to be present. To account for it, λ_i^v is determined as a function of time from the stability parameters, using a modified logarithmic decrement technique.[15] Using these values of λ_i^v to correct λ_i^*, the resulting damping factor λ_i is obtained. This correction process was used for all the nonlinear computations.

By employing Eqs. (14–19), the nonlinear stability coefficients were obtained by a least squares sectional fitting technique using 11 values of $K_{N,P}$, $\lambda_{N,P}$ and $\omega_{N,P}$ per section. The fits were incremented by one data point for each succeeding fit. The results, given in Table 1, show that the linear coefficient terms are consistent, whereas the nonlinear terms show slight oscillations. Average values of these coefficients are compared with the input for the numerical integration in Table 2. The linear terms were calculated with 3% errors, while the nonlinear restoring, damping and Magnus terms were calculated with errors of 13%, 11%, and 4%, respectively. Table 2 shows that the errors in the total stability coefficients at a 20° angle of attack are small (1–4%), indicating that $C_{M_\alpha}(|\alpha|)$, $C_{M_q}(|\alpha|) + C_{M_{\dot\alpha}}(|\alpha|)$ and $C_{M_{p\alpha}}(|\alpha|)$ can be determined accurately.

Effect of Random Error and Frequency Variation

For the purpose of testing the nonlinear theory on experimental type data, random error was introduced into the numerically integrated angular data for the basic aerodynamic coefficients. Errors were added to the α and β data, utilizing normally distributed random numbers. Two cases were in-

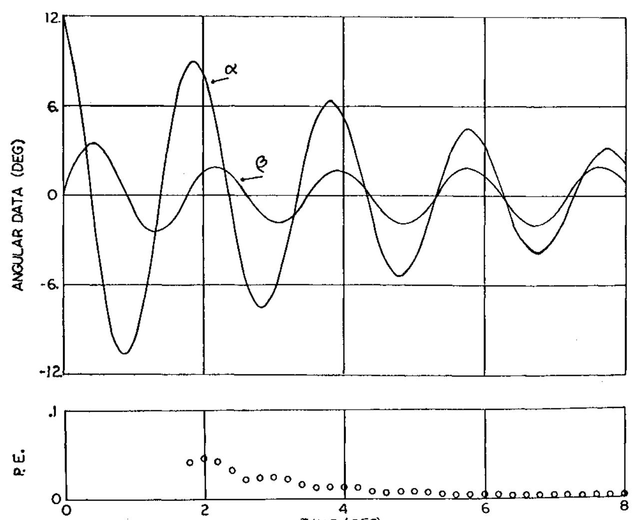

Fig. 2 Angles of attack and side slip, and probable error of fit for numerical simulation.

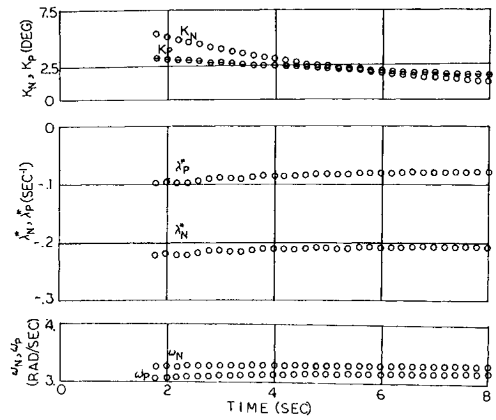

Fig. 3 Stability parameters from numerical simulation.

vestigated, one with maximum errors of ±0.25° and the other with ±1.0°. The angular data were then fitted, using the nonlinear aeroballistic reduction procedure, as discussed in the preceding sections. The average stability coefficient terms from the fit are given in Table 2. In general the nonlinear theory determined the linear terms of the stability coefficients accurately, but errors were large for the nonlinear damping (15–78%) and Magnus (40–112%) terms. However, the ±1° random error was considered extreme, as experimental data generally are determined to approximately ±0.25°. Thus, it was felt that the errors for the nonlinear terms would be acceptable.

As noted earlier, a frequency variation resulting from a nonlinear restoring moment will give an additional damping effect to the damping rates $\lambda_{N,P}^*$ and will introduce errors into the determination of $C_{M_q}(|\alpha|) + C_{M_{\dot\alpha}}(|\alpha|)$, and $C_{M_{p\alpha}}(|\alpha|)$. To show this effect, situations were investigated in which the frequency-variation correction was omitted. This analysis was carried out for the basic set of stability coefficients. The analysis procedure was the same as previously used. Table 2 shows that the nonlinear damping and Magnus terms were off by 52 and 45%, respectively. Thus, it is important to include the correction for frequency variation.

Linear Coefficients

As a further check on the value of the nonlinear theory, two cases were investigated using linear stability coefficients. In one, the restoring and Magnus coefficients were linear with respect to $|\alpha|$, and the damping coefficient was constant with $|\alpha|$. The input values and final fitted values are compared in the last four columns of Table 2, which indicates that the theory predicted $C_{M_{\alpha_2}}$ to be zero. The average values of both $(C_{M_q} + C_{M_{\dot\alpha_2}})$ and $C_{M_{p\alpha_2}}$ are not equal to zero; however, their effect on the total coefficient is small for $|\alpha| < 20°$. Also,

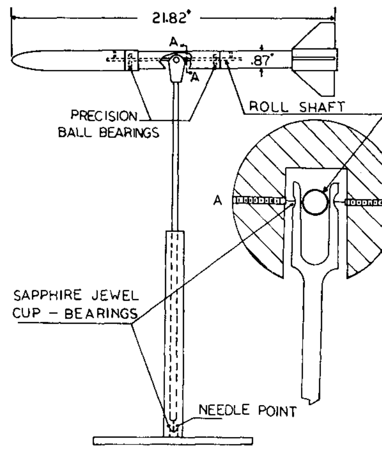

Fig. 4 Three-degree-of-freedom, jewel bearing support system for Apache rocket model.

the terms exhibited inconsistencies in the sectional fits as seen in the following case.

In the second case, a nonlinear restoring coefficient was used for numerical computations. The coefficient sectional fits are given in Table 3. It should be noted from these results that whenever a nonlinear term is zero, the computed values of this term in the sectional fits will vary in magnitude and sign. The input values and final fitted values are compared in Table 4. The errors for the restoring coefficient and the linear terms of the damping and Magnus coefficients are comparable to those of the previous studies.

Summary of Evaluation

This evaluation has indicated (for the range of aerodynamic data, initial conditions and mass parameters employed) that the nonlinear theory can determine the angle-of-attack variation of the restoring, damping and Magnus coefficients satisfactorily from a single set of data. This capability extends that of Ref. 10 which employed numerous sets of data and was not capable of determining the nonlinear damping.

The data reduction procedure will determine the linear coefficient terms quite accurately under most instances. However, the nonlinear coefficient terms must have a sufficient effect on the stability parameters, if they are to be accurately computed. If the nonlinear terms are small, this can be accomplished by having large initial angles of attack. From this technique, sign variations in the stability coefficients can easily be distinguished. When a nonlinear term is not present, the sectonal fitting will show a fluctuation in magnitude and/or change in sign for that particular term. Furthermore, the nonlinear terms tend to absorb the errors present in the angular data.

Table 2 Coefficient comparisons for means of basic data set, random error perturbations, frequency variations, and linear data

					Effects of random errors				No correction for frequency variation		With linear coefficients			
	Basic Set				± 0.25° Case I		± 1° Case II							
Coefficient	Input	Fit	e^a, %	e_t, %^{a} @$\|\alpha\|$ = 20°	Fit	e, %	Fit	e, %	Fit	e, %	Input	Fit	e, %	e_t, % @$\|\alpha\|$ = 20°
$C_{M_{\alpha_0}}$	−400	−413	3	0.8	−413	3	−410	2.5	−413	3	−400	−413	3	3
$C_{M_{\alpha_2}}$	1000	1126	13		1111	11	1015	1.5	1145	15	0	0		
$(C_{M_q} + C_{M_{\dot\alpha_0}})$	−10100	−10430	3	8.3	−10460	4	−10140	0.4	−10390	3	−10100	−10430	3	7
$(C_{M_q} + C_{M_{\dot\alpha_2}})$	−150000	−166700	11		−172800	15	−266500	78	−227800	52	0	−3392		
$C_{M_{p\alpha_0}}$	−600	−620	3	8.9	−638	6	−679	13	−632	5	−600	−620	3	5
$C_{M_{p\alpha_2}}$	6000	−6260	4		8379	40	12740	112	8709	45	0	−91		

[a] e = error; e_t = error in total coefficient.

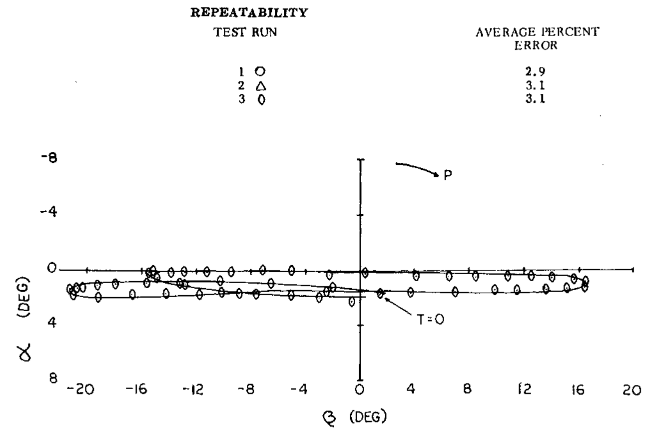

Fig. 5 Sample complex angle of attack and percentage errors of fit for test one.

Experimental Results

Present wind-tunnel testing procedures utilize primarily two distinct techniques: static testing, wherein strain-gage apparatus is used to measure the aerodynamic forces and moments directly, and dynamic testing, wherein the angular motions of delicately balanced models are measured. The latter approach is one employed at the University of Notre Dame. This technique was originally applied to one-degree-of-freedom motions,[18] and later was successfully extended to two- and three-degree-of-freedom angular motions.[19–21] The performance of the model support system is of vital importance in dynamic testing. In the past, ball bearings were used primarily in the supports, but subsequent analyses showed the results to be affected by bearing friction. Consequently, a sapphire jewel system (Fig. 4) was designed and has been shown to be effective in reducing frictional interference.[22,23,24] The model is mounted on needle points which are set into sapphire jewel cups. The sapphire jewels are synthetically fabricated, and their inherent hardness permits the needle points to rotate freely on their surfaces with a minimum of friction. Both pitch and yaw oscillations rely on these jewel cup bearings, while ball bearings were used for roll oscillations.

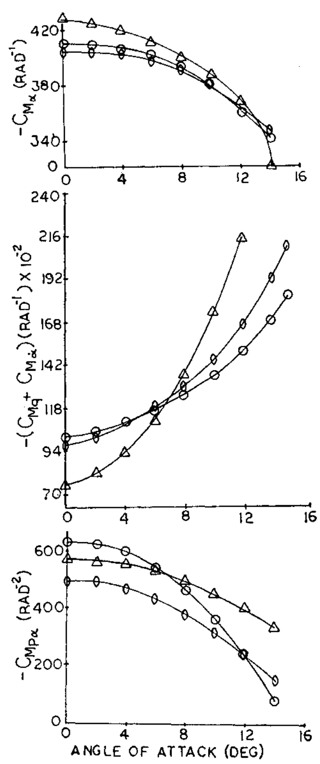

Fig. 6 Nonlinear subsonic aerodynamic stability coefficients for Apache sounding rocket.

Table 3 Coefficient sectional fits for data with constant damping, linear Magnus, and nonlinear restoring coefficients

T	$C_{M_{\alpha_0}}$	$C_{M_{\alpha_2}}$	$(C_{M_q} + C_{M_{\dot\alpha_0}})$	$(C_{M_q} + C_{M_{\dot\alpha_2}})$	$C_{M_{p\alpha_0}}$	$C_{M_{p\alpha_2}}$
2.75	−411.8	1038	−10020	−76540	−639.2	2132
2.85	−410.6	955	−10190	−47870	−641.5	1903
2.95	−410.5	940	−10470	7033	−632.7	624
3.05	−411.5	1004	−10720	57540	−618.8	−929
3.15	−413.0	1008	−10830	83770	−605.6	−2144
3.25	−414.3	1208	−10790	77820	−598.0	−2643
3.35	−415.1	1271	−10610	40840	−597.8	−2278
3.45	−415.1	1279	−10380	−12500	−605.1	−1181
3.55	−414.3	1222	−10180	−62430	−617.4	261
Mean	−412.9	1114	−10470	7485	−617.4	−472

Numerous 3-D wind-tunnel tests were conducted on the Apache model in the Notre Dame low-turbulence subsonic wind tunnel to establish the repeatability of the aerodynamic coefficients with angle of attack. An example of the α,β data obtained from one test is shown in Fig. 5. Because the model was designed with zero fin cant, it was necessary to add small fin tabs to induce rolling motions, and the resulting steady-state rolling velocity ranged in magnitude from 24–34 rad/sec. A wind-tunnel velocity of 38 fps was used for all the tests, and initial angular displacements of 10–12° were imposed. The model was initially disturbed, and its oscillations were recorded at 300–400 frames/sec by a Fastax motion picture camera, which viewed the model through a 45° mirror mounted at the rear of the test section. For use in the wobble program,[15] the angular data were digitized every 0.05 sec and these data were fitted in segments of 2.3 cycles of the elliptical motion, with each segment containing 190 points. The stability parameters, $K_{N,P}$, $\lambda_{N,P}$, $\omega_{N,P}$, were determined at a time interval of 0.1 sec. As indicated in Fig. 5, the average percentage error of fit of the epicyclic equation, Eq. (10), to the data was 3%.

Using Eqs. (10–19) the stability coefficients (Fig. 6) were determined as polynomial functions of the angle of attack, as defined by Eq. (8). Shown in Fig. 6 are the values of the coefficients obtained from the fits for three repeatability tests. The repeatability was good for the restoring coefficient, while the damping and Magnus coefficients exhibited a higher degree of scatter.

As predicted by Eqs. (4) and (14), the degree of damping for each mode is dependent on the relative signs and magnitudes of $C_{M_q} + C_{M_{\dot\alpha}}$ and $C_{M_{p\dot\alpha}}$. At low $|\alpha|$, the negative values obtained for both coefficients acted to stabilize the nutation mode K_N while destabilizing the precession mode K_p. At large $|\alpha|$ the Magnus coefficient $C_{M_{p\alpha}}$ was relatively small in magnitude, and hence the damping coefficient predominated, thus assuring damping of the precession mode, as well as the nutation model. As $|\alpha|$ decreased, however, $C_{M_{p\alpha}}$ assumed increasingly larger negative values, while the damping coefficient became smaller negatively, until a point was reached at

Table 4 Coefficient comparison for mean values from Table 3

Coefficient	Input	Fit	e, %	e_t, % @$\lvert\alpha\rvert = 20°$
$C_{M_{\alpha_0}}$	−400	−413	3	0.5
$C_{M_{\alpha_2}}$	1000	1114	11	
$(C_{M_q} + C_{M_{\dot\alpha_0}})$	−10100	−10470	4	5
$(C_{M_q} + C_{M_{\dot\alpha_2}})$	0	7485	—	
$C_{M_{p\alpha_0}}$	−600	−617	3	13
$C_{M_{p\alpha_2}}$	0	−472		

which the precession damping rate was zero. At that time the dynamic behavior of the Apache was characterized by a precession limit cycle. The nonlinear variations of the Magnus and damping coefficients are thus seen to be vital to the explanation of the Apache sounding rocket's observed wind-tunnel motion.

It should be noted that this analysis has been conducted for the condition of the roll rate approximately 8 times the nutation rate. Furthermore, the angular motion analyzed was pure epicyclic in nature. As a result the induced side moment of Nicolaides,[8,25] which applies only for the roll rate approximately equal to the nutation rate, was not considered applicable for the test conditions. The side moment postulated by Murphy et al.,[9,26,27] was not considered applicable, since this term applies for limit cycle motions, and these regions of the angular motion were avoided in the data analysis.

Conclusions

The nonlinear aeroballistic theory and data reduction procedure presented herein are designed to extract, from the angular motions of missiles, the aerodynamic stability coefficients for their restoring, damping and Magnus moments as nonlinear functions of the magnitude of the complex angle of attack. To evaluate this theory, typical nonlinear aerodynamic stability coefficients from wind-tunnel data were used for computer simulations to obtain the angle of attack and angle of side-slip as functions of time, and the reduction procedure was applied to this data. The errors in the total nonlinear restoring moment, and constant and linear portions of the damping and Magnus moments were small under most circumstances (3–9%), but the nonlinear portions of the damping and Magnus moments were not computed as precisely. Only in the cases of extreme random noise ($\pm 1°$) did these terms have unacceptable error.

The three-degree-of-freedom, low-friction sapphire jewel-bearing support system yielded repeatable results for the aerodynamic stability coefficients with angle of attack when applied to subsonic angular oscillations of a model of an Apache sounding rocket. The nonlinear theory represented the epicyclic motion to within a fitting accuracy of 3% and provided the nonlinear variation of the coefficients with angle of attack. The results demonstrated the vital importance of the Magnus coefficient in regard to dynamic stability of the Apache. The nonlinearity of this coefficient clearly affected the missile's damping characteristics and was responsible for the precession limit cycle observed in the wind-tunnel tests. Continued efforts are in progress to improve the accuracy of experimental techniques and extend the scope of the nonlinear theory.

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